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THE USE OF 2F/(I+F^*F) TO PROOF THE SPECTRAL THEOREM FOR UNBOUNDED SELF-ADJOINT OPERATORS

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Abstract

A number of proofs of the spectral theorem for unbounded self-adjoint operators in a complex Hilbert space have been developed. Most of them uses a bounded transform to get the desired results. In this paper, we proof the same theorem using a new a bounded self-adjoint operator transform constructed due to the mapping $z \mapsto 2z(1+z^2)^{-1}$ *for a complex number* z.

Keywords: *Unbounded operators, Self-adjoint operators, Spectral theorem.*

Introduction

A number of proofs for the spectral theorem for unbounded self-adjoint operators on a Hilbert space have been done by a number of scholar. In paper, we provide an alternative proof using the bounded transform of unbounded self-adjont operator. The operator is due to a bounded mapping $z \mapsto 2z(1+z^2)^{-1}$, for a complex number z, which is bounded above by 1 and below by -1. The function is defined on the whole of the real number set making it suitable to transform any unbounded operator defined on the whole of the real number space, $\mathbb{R}$.

We begin by highlighting basic results on densely closed operators and the spectral integral then construct a bounded operator due to the above mapping. Finally, the theorem is proved using its bounded version with reference to the bounded transform operator. In this paper, $\Im(f)$ implies an integral of f while the abbreviations Ran and Ker to refer to the range and kernel of an operator, respectively.

Definition: Closed Operator. Let $F : D(F) \subseteq \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be an operator. The operator is closed if for any convergent sequence $x_n \in D(F)$ such that $x_n \rightarrow x \in \mathbb{H}_1$, we have $Fx_n \rightarrow y \in \mathbb{H}_2$ implying that $Fx = y$.

The operator is densely defined if $\overline{D(F)} = \mathbb{H}$. An important property of densely defined closed operators is that they are equal to the adjoints of their adjoints (double adjoints), that is, $F = F^{**}$,**[4]**.

Definition: Graph of an operator. The graph of an operator F is the set $\Gamma_F = \{(x, Fx): x \in D(F)\}$.

The graph of an operator provides the best approach for analyzing unbounded operators because it has the sense of domain as well as the action of an operator. The graph is a subset of a product space, $\mathbb{H} \oplus \mathbb{H}$; most important, its adjoint brings about a decomposition on the very product space. If $\kappa: \mathbb{H}_1 \mapsto \mathbb{H}_2$ is function such that $\kappa(x,y) = (-y,x)$ for $x \in \mathbb{H}_1$ and $y \in \mathbb{H}_2$, then $\Gamma_{F^*} = \kappa(\Gamma_F)^{\perp}$. If the operator F is a closed operator, then $\mathbb{H} \oplus \mathbb{H} = \Gamma_{F^*} + \kappa(\Gamma_F)$, **[1,3,5]**.

An operator in said to be closed if its graph is a closed subspace of the product space $\mathbb{H} \oplus \mathbb{H}$. The adjoint operator F^*, too satisfies the conditions in the definition of a closed operator, hence, it is a closed operator.

This paper focuses on self-adjoint operators as such, we will be studying a special group of closed operators since self adjoint operators are closed by virtue of F being equal to F^*. It is vital that we highlight one of the basic properties of a densely linear defined operator, F, that, if its nullspace is $\{0\}$ and its range is dense in the Hilbert space, then, its adjoint is invertible and $(F^*)^{-1} = (F^{-1})^*$. When such operator is self-adjoint, then the relation becomes

$$F^{-1} = (F^*)^{-1} = (F^{-1})^*$$

implying that its inverse is also self-adjoint, [1,3].

The study of self-adjoint operators necessitates a time to time reference to symmetric operators, which in most cases, are regarded as their restrictions on a smaller space. An operator is F is symmetric if $D(F) \subseteq D(F^*)$ and $\langle Fu, v\rangle = \langle u, Fv\rangle$. Furthermore, the densely defined operator F is a symmetric operator, if and only if $\langle Fu, u \rangle$ is real for $u \in D(F)$$, **[5]**. If $D(F) = D(F^*)$ then F is self-adjoint. A times, it may be necessary to call self-adjoint operators as densely defined bounded symmetric operators, **[5]**.

The spectral theorem of self-adjoint operators is about the representation of a self-adjoint operator as an integral of an almost everywhere finite measurable function with respect to some unique projection values measure, the spectral measure, evaluated over its spectrum. As such, there is need to consider some important aspects of the spectral integrals.

Definition: Spectral measure. Let X be a set and $\mathfrak{X}$ its σ-aglebra, then the operator $P(\cdot)$ from $\mathfrak{X}$ to the Hilbert space $\mathbb{H}$, is a spectral measure if

1) $P(\theta)$ is an orthogonal projection, that is, $P^2(\theta) = P(\theta)$ and $P^*(\theta) = P(\theta), \theta \in \mathfrak{X}$

2) $P(X) = 1$,

3) $P(\bigcup_{i=1}^{\infty} \theta_i) = \sum_{i=1}^{\infty} P(\theta_i)$ for $\theta_i \in \mathfrak{X},\ i \in \Lambda$ such that $\theta_i \cap \theta_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} \theta_i = X$.

Definition: Spectral integral. Let X be a set, $\mathfrak{X}$ its σ-aglebra, $\mathcal{W} = (X, \mathfrak{X}, P)$ be a space of P-almost everywhere $(a.e)$ −finite measurable function $\Psi: X \to \mathbb{C} \cup \infty$, where P is a spectral measure. Let $(M_i)_{i\in\mathbb{N}}$ be sequences of sets in $\mathfrak{X}$ such that $M_i \subseteq M_{i+1}$ and $P(\bigcup_{i=1}^{\infty} M_i) = I$ where

$$M_i = \{z \in X : \ \|\Psi_{\mathrm{k}}(z)\| \leq \ n\,, k = 1,2, \dots, m\}$$

for all Ψ. Then the spectral integral (for unbounded measurable function) is

$$\mathfrak{I}(\Psi) = \operatorname*{Sup}_{i\in\mathbb{N}} \int_X \Psi \chi_{M_i}(z)\, \mathrm{d}P(z)$$

where χ_{M_i} is an indicator function on M_i. This integral $\mathfrak{I}(\Psi)$ is an operator and its domain is

$$D\big(\mathfrak{I}(\Psi)\big) = \{x : \ \|\mathfrak{I}(\Psi)x\|^2 < \infty \ \text{ for } \ x \in \ \mathbb{H}\,\}$$

where

$$\|\mathfrak{I}(\Psi)x\|^2 = \operatorname*{Sup}_{i\in\mathbb{N}} \int_X \left|\Psi \chi_{M_i}(z)\right|^2 \mathrm{d}\langle P(z)x, x\,\rangle \ < \infty.$$

The properties of the general integral $\mathfrak{I}(\Psi)$as well as $\mathfrak{I}(\Psi) \geq 0$can be found in **[5, 1]**. We now proceed to the properties of our transform.

2. Development of the transform

In this section, we developing the transform operator due to the mapping $z \mapsto 2z(1+z^2)^{-1}$ for a complex number z. To define an inverse for the function, we break it into two parts, on $[-1,1]$ and on $\mathbb{R}\backslash[-1,1]$. The inverse of the function $s(z) = 2z(1+z^2)^{-1}$ on $[-1,1]$ is $f_1(w) = \frac{1-\sqrt{1-w^2}}{w}$ while that of the same function on $\mathbb{R}\backslash[-1,1]$ is $f_1(w) = \frac{1+\sqrt{1-w^2}}{w}$. We discuss important properties of the operator due to the function $s(z)$ that will be useful in the proof of the spectral theorem for the unbounded operators.

Lemma 2:1

Let $F : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be a densely defined linear closed operator between two Hilbert spaces, then

1) The operator $I + F^*F$ is a bijective hence invertible

2) The inverse operator $(I + F^*F)^{-1}$ is a bounded linear self-adjoint operator, and $0 \leq (I + F^*F)^{-1} \leq I$.

3) The operator $I + F^*F$ is self-adjoint so is F^*F.

Proof

(1). Since F is a closed densely defined operator whose graph is Γ_F, we have $\mathbb{H}_1 \oplus \mathbb{H}_2 = \Gamma_{F^*} \oplus \kappa(\Gamma_F)$. Therefore, for every $h \in \mathbb{H}_1$, there exists $u \in D(F)$ and $v \in D(F^*)$ such that

$$(0, h) = (v, F^*v) + \kappa(u, Fu)$$

$$= (v, F^*v) + (-Fu, u)$$

$$= (v - Fu, F^*v + u).$$

Thus, $0 = v - Fu$ and $h = F^*v + u$ implying that $v = Fu$ and $h = u + F^*v = u + F^*Fu = (I + F^*F)u$. Thus, for each $h \in Ran(I + F^*F)$, we have $u \in D(I + F^*F)$ implying that $I + F^*F$ is surjective.

On the other hand, we have

$$\|(I + F^*F)u\|^2 = \langle u + F^*Fu, u + F^*Fu \rangle$$

$$= \|u\|^2 + \|F^*Fu\|^2 + \langle u, F^*Fu\rangle + \langle F^*Fu, u \rangle$$

$$= \|u\|^2 + \|F^*Fu\|^2 + \langle Fu, Fu\rangle + \langle Fu, Fu\rangle$$

$$= \|u\|^2 + \|F^*Fu\|^2 + 2\|Fu\|^2$$

$$\|(I + F^*F)u\| = \|u\|.$$

Thus, $I + F^*F$ is bounded below implying that $Ker(I + F^*F) = \{0\}$, therefore, $I + F^*F$ is injective. The bijectivity of $I + F^*F$ implies that it is invertible, thus, $(I + F^*F)^{-1}$ exists.

(2). Let $u \in D(I + F^*F) = D(I) \cap D(F^*F) = D(F^*F)$, then there is $h \in Ran(I + F^*F)$ such that $h = (I + F^*F)u$. Since $I + F^*F$ is invertibe, we have $u = (I + F^*F)^{-1}h$. Furthermore, for $u \in D(F^*F)$, $\langle F^*Fu, u\rangle = \langle Fu, Fu\rangle = \|Fu\|^2 \geq 0$. Therefore,

$$\|(I + F^*F)^{-1}h\| = \|Iu\| \leq \|(I + F^*F)u\| = \|h\|$$

Thus, $0 \leq (I + F^*F)^{-1} \leq I$ implying that it is bounded above.

Next, we show that $(I + F^*F)^{-1}$ is symmetric. Denote $(I + F^*F)^{-1}$ by L hence

$$\begin{aligned}\langle Lh, h\rangle &= \langle (I + F^*F)^{-1} h, h\rangle \\ &= \langle u, h\rangle \\ &= \langle u, u + F^*Fu\rangle \\ &= \langle u, u\rangle + \langle u, F^*Fu\rangle \\ &= \langle u, u\rangle + \langle Fu, Fu\rangle \\ &= \|u\|^2 + \|Fu\|^2 \in \mathbb{R}.\end{aligned}$$

Thus, $L = (I + F^*F)^{-1}$ is symmetric. Since $(I + F^*F)^{-1}$ is symmetric and bounded, it is self-adjoint.

(3). The operator $L = (I + F^*F)^{-1}$ is self-adjoint hence closed. It is invertible and its inverse is $I + F^*F$. Thus, $Ker(L) = \{0\} = Ran(L)^{\perp}$. This implies that $Ran(L)$ is dense in $\mathbb{H}_2$ hence the inverse of L^* exists and

$$L^{-1} = (L^*)^{-1} = (L^{-1})^*.$$

Thus, the inverse is self-adjoint, **[1, 5]**.

The fact that F^* is closed and densely defined linear operator, we have $F = F^{**}$, hence applying adjoint twice on the following

$$F^*F \subseteq F^*F^{**}$$

we get that $F^*F = (F^*F)^*$ thus, F^*F is self-adjoint. □

We now investigate the operator $S = 2FL$ where $L = (I + F^*F)^{-1}$ and F is a densely defined closed operator on a Hilbert space, $\mathbb{H}$. .

Lemma 2.2

Let F be a densely defined closed linear operator on a Hilbert space, $\mathbb{H}$ then,

1) The operator S is bounded and $\|S\| \leq 1$.

2) The operator S is self-adjoint if and only if F is self-adjoint.

Proof

Let $x, y \in D(F^*F) = D(I + F^*F)$, and $u, v \in Ran(I + F^*F)$ be such that $u = (I + F^*F)x$ and $v = (I + F^*F)y$ then $Lu = x$ and $Lv = y$. The operator $L = (I + F^*F)^{-1}$$ is bounded and $\|L\| \leq 1$, hence $\|L^* - L^*L\| \leq \frac{1}{4}$. From the relation $L(1 + F^*F) = I$, we have $Lu + F^*FLu = u$, therefore

$$\|Su\|^2 = \|2FLu\|^2$$

$$= 4\langle Lu, F^*FLu \rangle$$

$$= 4\langle Lu, u - Lu \rangle$$

$$= 4\langle u, L^*u - L^*Lu \rangle$$

$$\leq 4\langle u, \mathrm{Sup}(L^* - L^*L)u \rangle$$

$$= 4\langle u, 0.25u\rangle$$

$$= \langle u, u\rangle$$

$$= \|u\|^2$$

$$\|Su\| \leq \|u\|.$$

This implies that $\|S\| \leq 1$.

(2). From above conditions together with the relation $L(I + F^*F) = I$ which implies that

$LF^* + LF^*FF^* = F^*$ and the self-adjointness of F^*F, we have

$$\langle 2FLu, v\rangle = 2\langle 2FLu, (I + F^*F)y\rangle$$

$$= 2\langle Lu, F^*(I + F^*F)y\rangle$$

$$= 2\langle Lu, F^*(I + F^*F)Lv\rangle$$

$$= 2\langle u, (LF^* + LF^*F^*F)Lv\rangle$$

$$= 2\langle u, (LF^* + LF^*FF^*)Lv\rangle$$

$$= 2\langle u, F^*Lv\rangle$$

$$= \langle u, 2F^*Lv\rangle$$

$= \langle u, 2FLv \rangle,$

only if F is symmetric. Thus, $S = 2FL$ is symmetric if and only if F is symmetric. Since $S = 2FL$ is bounded, then $S = 2FL$ is self-adjoint if and only if F is self-adjoint.

3. Spectral theorem for the unbounded self-adjoint operators

We now proceed to state and prove the spectral theorem for unbounded self-adjoint operator using the bounded transform $S = 2FL$.

Theorem 3.1

Given that F is a self-adjoint operator on a Hilbert space $\mathbb{H}$, then there exists a unique spectral measure P_F, dependent on F\$ on the Borel sigma-algebra $B(\mathbb{R})$ such that

$$F = \int_{\mathbb{R}} z \, \mathrm{d}P_F(z) \, .$$

Proof

From Lemma 2.2, the operator $S = 2FL = 2F(I + F^*F)^{-1}$ is a bounded self-adjoint operator on a Hilbert space $\mathbb{H}$ if F is a self-adjoint operator. Furthermore, $\|S\| \leq 1$ implying that $\sigma(S) \subseteq [-1,1]$ where $\sigma(S)$ denotes the spectrum of the operator S.

Let us denote the set $\mathfrak{T} = [-1,1]$, then $\mathfrak{T}$ is compact interval on $\mathbb{R}$. If $B(\mathfrak{T})$ is the Borel sigma algebra of $\mathfrak{T}$, then by the spectral theorem of bounded self-adjoint operators **[5]**, there exists a unique spectral measure P_F, dependent on F, on $B(\mathfrak{T})$ such that

3.1 $$S = \int_{\mathfrak{T}} w \, \mathrm{d}P'_S(w) \, .$$

Thus, $S = \mathfrak{I}(w)$. Since S is invertible (piecewise) we have $Ker(S) = \{0\}$. Consequently, $P'(\{0\})\mathbb{H} \subseteq Ker(S) = \{0\}$ implying that $P'(\{0\}) = 0$. Therefore, $f(w) = \frac{1 \pm \sqrt{1-w^2}}{w}$ are $P' - a.e$ finite Borel functions on $\mathfrak{T}$.

The domain of the integral operator $\mathfrak{I}$, of the function $f(w)$ is

$$D\big(\mathfrak{I}\,(f)\big) = \left\{x \in \mathbb{H} : \int |f(w)|^2 \, \mathrm{d}\langle P'(w)x, x\rangle \, < \infty \right\}.$$

Given the relation $S = \mathfrak{I}(w)$, we seek to proof that $F = \mathfrak{I}(f)$ given that $S = \mathfrak{I}(w)$. We have $|w| \leq 1$, hence $1 - w^2 \geq 0$. Using properties of integrals in **[1, 5]**, we apply integrals on $f(w) = \frac{1 \pm \sqrt{1-w^2}}{w}, w \neq 0$ to get

$$\mathfrak{I}(f(w)) = \mathfrak{I}\left(\frac{1 \pm \sqrt{1-w^2}}{w}\right) = \frac{1 \pm \sqrt{1 - \mathfrak{I}(w^2)}}{\mathfrak{I}(w)} = \frac{1 \pm \sqrt{1 - S^2}}{S}.$$

We have two cases $\Im\big(f(w)\big) = \frac{1+\sqrt{1-S^2}}{S}$ and $\Im\big(f(w)\big) = \frac{1-\sqrt{1-S^2}}{S}$ which are equivalent to $\Im(f) - \frac{1}{S} - \frac{\sqrt{1-S^2}}{S} = 0$ and $\Im(f) - \frac{1}{S} + \frac{\sqrt{1-S^2}}{S} = 0$, therefore,

$$\left(\Im(f) - \frac{1}{S} - \frac{\sqrt{1-S^2}}{S}\right)\left(\Im(f) - \frac{1}{S} - \frac{\sqrt{1-S^2}}{S}\right) = 0$$

$$\Im^2(f) - \frac{2\Im(f)}{S} + \frac{1}{S^2} - \frac{1-S^2}{S^2} = 0$$

$$\frac{S^2\Im^2(f) - 2S\Im(f) + 1 - 1 + S^2}{S^2} = 0$$

$$S^2\Im^2(f) - 2S\Im(f) + S^2 = 0$$

$$S\Im^2(f) - 2\Im(f) + S = 0$$

$$S(I + \Im^2(f)) - 2\Im(f) = 0$$

$$S = \frac{2\Im(f)}{I + \Im^2(f)}$$

$$= \frac{2F}{I + F^*F}.$$

We have that $\Im(f) = F$.

We now prove the existence of a unique spectral measure P' on $\mathfrak{T}$. To prove the uniqueness of the spectral measure in the unbounded case, it is enough to show the same in the bounded case due to the bounded transform $S = 2F(I + F^*F)^{-1}$. Since $w \neq 0$, we can define an inverse from w $[-1,1] \setminus \{0\}$ to $f(w) = z \in \mathbb{R}$.

We defined f as $f(w) = \frac{1\pm\sqrt{1-w^2}}{w}$ where the inverse of $f(w) = \frac{1-\sqrt{1-w^2}}{w}$ was $f^{-1}(z) = \frac{2z}{1+z^2}$ for $z \in [-1,1]$ and that of $f(w) = \frac{1+\sqrt{1-w^2}}{w}$ was $f^{-1}(z) = \frac{2z}{1+z^2}$ for $z \in \mathbb{R}\setminus[-1,1]$. Let f be a function from the measure space $(\mathfrak{T}, B(\mathfrak{T}), P')$ to $(\mathbb{R}, B(\mathbb{R}), P)$ where P is the spectral measure on $B(\mathbb{R})$.

For unique mapping, let $[-1, a] = \chi \subset B(\mathfrak{T})$ for $-1 < a \leq 0$ and $\mathbb{R}^-$ denote the negative part of the real number system, we define the inverse function f as $f^{-1}\big(\eta_2 - (\mathbb{R}^- \setminus[-1,1] - \eta_1)\big) = \chi$ for $\eta_1, \eta_2 \subseteq B(\mathbb{R})$ where $(-\infty, b] = \eta_1 \subseteq (-\infty, -1] \subset \eta_2 \subseteq (-\infty, 0]$ for $-\infty < b \leq -1$. The function is well defined since $B(\mathbb{R})$ is a sigma -algebra.

On the other hand, let $[-1, a] = \chi \subset B(\mathfrak{T})$ for $0 < a \leq 1$, we define the inverse function f as $f^{-1}(\eta_1 \cup (\mathbb{R} - \eta_2)) = \chi$ for $\eta_1, \eta_2 \subseteq B(\mathbb{R})$ where $(-\infty, b] = \eta_1 \subseteq (-\infty, 1] \subset \eta_2 \subseteq \mathbb{R}, 0 < b \leq 1$.

Likewise, the function is well defined since $B(\mathbb{R})$ is a sigma -algebra. Letting θ be any of above arguments for f^{-1}, we have

$$P(\theta) = P'(f^{-1}(\theta))$$

is a measure on $B(\mathbb{R})$. Therefore, by the push-forward integral formula **[2]**, the integral of z is given by

$$\int_{\mathbb{R}} z \, \mathrm{d}P(z) = \int_{\mathfrak{T}} f(w) \, \mathrm{d}P'(w) = \mathfrak{J}(f) = F.$$

To prove the uniqueness, the apply the assertion of the uniqueness of spectral measure for bounded self-adjoint operators. Given that $\mathcal{P}$ is another spectral measure on elements of $B(\mathbb{R})$ such that

$$F = \int_{\mathbb{R}} z \, \mathrm{d}\, \mathcal{P}_F(z) \ .$$

Then $\mathcal{P}(\chi) = \mathcal{P}'(f(\chi))$ for $\chi \in B(\mathfrak{T}\backslash\{0\})$ is also spectral measure, hence

$$S = 2FL = \int_{\mathbb{R}} f^{-1}(z) \, \mathrm{d}\mathcal{P}(z) = \int_{\mathfrak{T}\backslash\{0\}} w \, d\mathcal{P}'(w) \, .$$

Define $\mathcal{P}'(\chi)$ as $\mathcal{P}'(\chi) := \mathcal{P}'(\chi \cap (\mathfrak{T}\backslash\{0\})), \chi \in B(\mathfrak{T})$, then $\mathcal{P}'(f(\chi))$ is extended to $\mathfrak{T}$ by $\mathcal{P}'(\chi)$ defined above, thus, by the uniqueness assertion of the spectral theorem for bounded self-adjoint operators, we get that $\mathcal{P}' = P'$ and $\mathcal{P} = P$ as required.

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